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A classical description of spinning particles

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Abstract. On the basis of a calculation for the force and couple exerted on a moving point dipole given previously, a consistent description of classical spin for particles possessing charge and dipole moment is discussed.

1. Introduction

One of the difficulties associated with the classical treatment of spin for particles possessing charge and dipole moment lies in the variety of possible equations which may be used satisfactorily to describe it. Thus we find many differing interpretations of classical spin with no overriding treatment which is fundamental. Although the latitude of choice for dynamical expressions representing force and couple, consistent with relativistic particle mechanics, is small, there has been no general agreement as to the exact expressions which should be used to represent the effect produced by an external electromagnetic field on a moving dipole. While it is known that for many applications of classical spin, and also for reasons of simplicity, an exact knowledge of detail is not necessary, it would be useful to find a complete picture describing every possibility and which could with advantage be used to describe, for example, the classical evaluation of the spin gyromagnetic ratio.

In a recent article (Ellis 1970) certain expressions were derived for the force and couple exerted on a moving electric dipole, which were relativistically covariant and which led to the usual classical expressions for the force and couple for a stationary dipole. The method involved in obtaining these expressions was based on an invariant action principle which led to equations of motion containing the expressions for the force and couple. The validity of the results obtained for these quantities was independently established, in the case of constant velocity, by agreement with the transformed expressions by Lorentz transformation from the usual static formulae. The particular model used to describe the dynamical aspects of the motion, and in particular that for the spin angular momentum, was not pursued in detail in that work and comparison with standard treatments of spin was not made. In view of the known validity of the expressions for the force and couple, it seems worthwhile to investigate further the dynamical equations from which the results were derived, and we hope to show in the present article that the model chosen possesses several advantages over other models of a more conventional type, and therefore provides to some extent a justification of the equations given previously, and some new lines of approach in the description of classical spin.

A fact which arises from the equations given previously, in the case of constant spin, is that if we impose the condition that the rest mass of the pole–dipole particle remains constant throughout its motion, it follows (apart from one or two possible and perhaps less dynamically interesting alternatives) that the spin angular momentum of the particle should remain parallel to the dipole moment (electric or magnetic). It is clear that the imposition of a condition of this kind on relativistically covariant equations must lead to a restriction of some kind on the motion of the particle, and the fact that this alignment of spin and moment (often assumed in many classical arguments) arises naturally, is quite useful. We devote some time in § 2 to discussing this and other possibilities when the equations are taken in their general form.

A model given in the book by Rohrlich (1965) which is an improvement of the treatment given by Panofsky and Phillips (1964) to account classically for the spin gyromagnetic ratio ($g = 2$) is one in which it is assumed that the spin is parallel to the magnetic moment, and we show that in this case, where the magnitude of the spin is assumed constant, the form of his equations (the space components of which reduce in the nonrelativistic limit to the classical equation $\mathrm{d}\boldsymbol{s}/\mathrm{d}t = \boldsymbol{m} \times \boldsymbol{B}$) agree with, and is a solution of, our own. Although Rohrlich's treatment assumes that the conventional condition of spin, $S^{\mu\nu}V_\nu = 0$, is satisfied, it is apparent that, in an instantaneous rest-frame an interpretation is lacking for the nonspatial components of his equations. However, his treatment may be satisfactorily described by means of our own equations.

The equation for the force, which now becomes modified by the addition of a dipole force, cannot now be used to deduce the gyromagnetic ratio as was the aim in the treatments of Rohrlich and Panofsky and Phillips, but an essentially similar equation exists in component form which may be used directly to obtain a relationship between the magnitudes of spin and moment. This leads to a slightly modified value for the gyromagnetic ratio which, if taken as accurate, leads to a credible relationship between the rest energy and dipole energy of the particle.

Returning to the equations in their general form we investigate in § 4 the consequences arising from dropping the condition that the spin magnitude is constant, but the solutions of dynamical importance as compared with § 2 are found to be those where spin angular momentum and angular velocity of the particle are in line. Such particles we call 'spherical' type particles.

In § 5 it is shown how, through the canonical momentum

$$p^\mu = G^\mu + eA^\mu + q_\alpha F^{\mu\alpha} - m_\alpha \overset{*}{F}{}^{\mu\alpha}$$

containing dipole moment terms, the transition to a relativistic quantum theory may be achieved in the case of constant spin, necessitating only the 'minimal' coupling

$$\partial^\mu \rightarrow \partial^\mu + \mathrm{i}(e/\hbar c)A^\mu$$

and a small change in the magnitude of the rest mass. The relevance of Rohrlich's equations to this is shown.

The arguments here given are in the nature of mathematical proof and therefore we shall use covariant description throughout, although in one or two cases equations have been written in terms of three-dimensional vectors to simplify their understanding.

Antisymmetric tensors play an important part in classical descriptions of spin and we have found it useful to split such quantities into their respective vector components. This approach is unlike any methods hitherto used in classical treatments of spin and the method does not lose any generality. For the field and dipole moments (evaluated at the particle's position) we shall write, without loss of generality

$$\begin{aligned} F^{\mu\nu} &= -(E^\mu V^\nu - E^\nu V^\mu) + \xi^{\mu\nu\alpha\beta} B_\alpha V_\beta \\ p^{\mu\nu} &= (q^\mu V^\nu - q^\nu V^\mu) + \xi^{\mu\nu\alpha\beta} m_\alpha V_\beta \end{aligned} \tag{1.1}$$

where the electric and magnetic vector components of the field (E^μ and B^μ) and of dipole moment (q^μ and m^μ) are orthogonal to the four-velocity of the particle V^μ, and therefore represent the quantities actually measured by an observer travelling with the particle. From conventional electromagnetic theory we may say that we should expect the force and couple measured by such an observer would contain the terms

$$\begin{aligned} f^\mu &= eE^\mu + q^\alpha \partial_\alpha E^\mu + m^\alpha \partial_\alpha B^\mu \\ C^{\mu\nu} &= -(q^\mu E^\nu - q^\nu E^\mu) - (m^\mu B^\nu - m^\nu B^\mu) \end{aligned} \tag{1.2}$$

corresponding to the usual expressions

$$\begin{aligned} \boldsymbol{f} &= e\boldsymbol{E} + (\boldsymbol{q}\,.\nabla)\boldsymbol{E} + (\boldsymbol{m}\,.\nabla)\,\boldsymbol{B} \\ \boldsymbol{C} &= \boldsymbol{q}\times\boldsymbol{E} + \boldsymbol{m}\times\boldsymbol{B} \end{aligned}$$

for the force and couple. These terms, together with other terms $\dot{\boldsymbol{q}}\times\boldsymbol{B} - \dot{\boldsymbol{m}}\times\boldsymbol{E}$ arising from the current effects of the motion of the poles of the dipole in their reaction with the field, agree with the valid expressions for the force and couple referred to earlier, as may easily be established.

2. The equations of motion

We now consider in detail the equations derived previously for the motion of a pole–dipole particle in the presence of an external electromagnetic field. We shall use the same notation as in the previous article (Ellis 1970), and our first aim is to discover the consequences arising from the condition of constant rest mass as referred to in the introduction. We shall first restrict the argument to the case where the magnitudes of the spin angular momentum and dipole moments of the particle remain constant, since the task involved is easier, but later (in § 4) we shall remove these conditions, and the additional terms arising in the equations of this section will be indicated.

The expressions for the four-force f^μ and couple six-vector $C^{\mu\nu}$ exerted on a point particle, possessing charge e and electric and magnetic dipole four-moments q^μ, m^μ and travelling with four-velocity V^μ in an external electromagnetic field $F^{\mu\nu}$ were previously found to be†

$$f^\mu = -eF^{\mu\nu}V_\nu - \dot{q}_\alpha F^{\mu\alpha} - q_\alpha F^{\mu\nu,\alpha}V_\nu + \dot{m}_\alpha \overset{*}{F}{}^{\mu\alpha} + m_\alpha \overset{*}{F}{}^{\mu\nu,\alpha}V_\nu \tag{2.1}$$

$$C^{\mu\nu} = 2q^{[\mu}F^{\nu]\alpha}V_\alpha - 2m^{[\mu}\overset{*}{F}{}^{\nu]\alpha}V_\alpha. \tag{2.2}$$

(The terms arising from the magnetic dipole moment m^μ may be obtained by the replacement of q^μ by $-m^\mu$ and $F^{\mu\nu}$ by $\overset{*}{F}{}^{\mu\nu}$.) The validity of these expressions was illustrated in the case of constant velocity, and their validity for all nonconstant values of V^μ will be assumed. It is straightforward to illustrate that the terms given in (1.2) are contained in these expressions.

The dynamical equations governing the motion of the particle:

$$\dot{G}^\mu = f^\mu \tag{2.3}$$

$$\dot{S}^{\mu\nu} + 2G^{[\mu}V^{\nu]} = C^{\mu\nu} \tag{2.4}$$

† These equations were originally given in a different article (Ellis 1966).

(where G^{μ} is the mechanical four-momentum and $S^{\mu\nu}$ is the spin angular momentum six-vector for the particle, connected by $G^{\mu} = m_0c^2V^{\mu} - \dot{S}^{\mu\nu}V_{\nu}$) were found from a general Lagrangian approach, and this formed the basis of our arguments leading to the construction of f^{μ} and $C^{\mu\nu}$. It is not asserted that these equations should be regarded as the only possible equations governing classical spin, but their simple form and their independence of any electromagnetic quantity are significant (see, for example, Barut 1964 pp. 83–84, Rohrlich 1965 p. 208 for references to other dynamical models).

Combining (2.1), (2.3) and (2.2), (2.4) we now impose the condition of constant rest mass $\dot{m}_0 = 0$. We have, from $m_0c^2 = G^{\mu}V_{\mu}$, the equation

$$\dot{G}_{\mu}V^{\mu} + G_{\mu}\dot{V}^{\mu} = 0.$$

From (2.1), (2.3) and the expression for G^{μ} this becomes:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) \equiv -\dot{q}^{\alpha}F_{\mu\alpha}V^{\mu} + \dot{m}^{\alpha}\overset{*}{F}_{\mu\alpha}V^{\mu} - \dot{S}_{\mu\nu}V^{\nu}\dot{V}^{\mu} = 0 \tag{2.5}$$

other terms having vanished identically.

Since the four-moment q^{α} has constant norm ($\dot{q}^{\alpha}q_{\alpha} = 0$) and is rotating about the world line, we may write

$$\dot{q}^{\alpha} = \underset{q}{\Omega}{}^{\alpha}{}_{\beta}q^{\beta} \qquad (\underset{q}{\Omega}{}^{\alpha}{}_{\beta} = -\underset{q}{\Omega}{}_{\beta}{}^{\alpha}). \tag{2.6}$$

If q^{α} does not rotate, then the antisymmetric angular velocity tensor $\underset{q}{\Omega}{}^{\alpha}{}_{\beta}$ reduces to that for the Thomas precession:

$$\underset{q}{\Omega}{}^{\alpha}{}_{\beta} = \underset{\mathrm{T}}{\Omega}{}^{\alpha}{}_{\beta} \equiv \dot{V}^{\alpha}V_{\beta} - \dot{V}_{\beta}V^{\alpha}$$

and q^{α} is then simply transported along the world line by the equations (of Fermi–Walker propagation):

$$\dot{q}^{\alpha} = -(\dot{V}_{\beta}q^{\beta})V^{\alpha}$$

(see, for example, Panofsky and Phillips 1964 p. 441 eqn (23–72)). However, in general when q^{α} rotates, we include this term in the expression for $\underset{q}{\Omega}{}^{\alpha\beta}$:

$$\underset{q}{\Omega}{}^{\alpha\beta} = \underset{\mathrm{T}}{\Omega}{}^{\alpha\beta} + \frac{1}{c}\xi^{\alpha\beta\mu\nu}\underset{q}{\omega}{}_{\mu}V_{\nu} \tag{2.7}$$

where $\underset{q}{\omega}{}^{\alpha}$ is the angular velocity four-vector representing the rate of rotation of q^{α}, such that $\underset{q}{\omega}{}^{\alpha}V_{\alpha} = 0$. (In the instantaneous rest-frame $\underset{q}{\omega}{}^{\alpha}$ has components $\underset{q}{\omega}{}^{\alpha} \equiv (0, \underset{q}{\boldsymbol{\omega}})$.) The form of (2.7) arises unambiguously from the general form of an antisymmetric tensor, the two 'vector' parts being $\dot{V}^{\alpha}$ and $\underset{q}{\omega}{}^{\alpha}$, both orthogonal to V^{α} (and therefore $\underset{q}{\Omega}{}^{\alpha\beta}$ contains its correct six components). The first vector part ensures that rotation takes place in the orthogonal three-space ($q^{\alpha}V_{\alpha} = 0$), since we have from (2.6) and (2.7)

$$\dot{q}^{\alpha} = -(\dot{V}_{\beta}q^{\beta})V^{\alpha} + \frac{1}{c}\xi^{\alpha\beta\mu\nu}q_{\beta}\underset{q}{\omega}{}_{\mu}V_{\nu} \tag{2.8}$$

and therefore

$$\dot{q}^{\alpha}V_{\alpha} + q^{\alpha}\dot{V}_{\alpha} = 0.$$

Using (2.8) and an analogous expression for $\dot{m}^{\alpha}$ we have for the first two terms of (2.5) the following:

$$-\dot{q}^{\alpha}F_{\mu\alpha}V^{\mu}+\dot{m}^{\alpha}\overset{*}{F}_{\mu\alpha}V^{\mu} = -\frac{1}{c}\xi^{\alpha\beta\sigma\tau}q_{\beta}\underset{q}{\omega}_{\sigma}V_{\tau}F_{\mu\alpha}V^{\mu} + \frac{1}{c}\xi^{\alpha\beta\sigma\tau}m_{\beta}\underset{m}{\omega}_{\sigma}V_{\tau}\overset{*}{F}_{\mu\alpha}V^{\mu} \tag{2.9}$$

where $\underset{m}{\omega}^{\alpha}$ is the angular velocity four-vector representing the rate of rotation of m^{α}, such that $\underset{m}{\omega}^{\alpha}V_{\alpha} = 0$. (We note that $\underset{\mathrm{T}}{\Omega}^{\alpha\beta}$ does not contribute in these terms.)

We now consider the final term of (2.5). The spin angular momentum tensor $S^{\mu\nu}$ is conventionally chosen to have its components orthogonal to the worldline and is therefore similar to the electromagnetic moment tensor of a magnetic dipole, so that we write

$$\frac{1}{c}S_{\mu\nu} = \xi_{\mu\nu\alpha\beta}s^{\alpha}V^{\beta} \tag{2.10}$$

where s^{α} is the spin angular momentum four-vector of the particle, orthogonal to the worldline. (This is entirely equivalent to the so-called 'Weyssenhoff condition'.) The components of $S_{\mu\nu}$ are therefore given by

$$\left(\frac{1}{c}S_{23},\frac{1}{c}S_{31},\frac{1}{c}S_{12}\right) \equiv -\beta\boldsymbol{s} \qquad \left(\frac{1}{c}S_{01},\frac{1}{c}S_{02},\frac{1}{c}S_{03}\right) \equiv \beta\boldsymbol{s}\times\frac{\boldsymbol{V}}{c}$$

where $\boldsymbol{s}$ represents the 'observed' spin (cf. Ellis 1970 equation (2.12)).

It is clear that whether the norm of the spin angular momentum $(-s^{\alpha}s_{\alpha})^{1/2}$ is constant, or not, the derivative of (2.10) with respect to the proper time, when multiplied by $V^{\nu}\dot{V}^{\mu}$, gives zero. Hence the equation (2.5) reduces by (2.9) to the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) \equiv \frac{1}{c}\xi^{\alpha\beta\sigma\tau}V_{\tau}(q_{\beta}\underset{q}{\omega}_{\sigma}F_{\alpha\mu}V^{\mu}-m_{\beta}\underset{m}{\omega}_{\sigma}\overset{*}{F}_{\alpha\mu}V^{\mu}) = 0. \tag{2.11}$$

We may simplify the terms in parentheses in (2.11) with the help of the equations of motion (2.2), (2.4), if it is now assumed that the particle possesses only an electric dipole moment ($m_{\beta} = 0$) or only a magnetic dipole moment ($q_{\beta} = 0$) or both electric and magnetic dipole moments inclined to each other at a constant angle ($q^{\alpha}m_{\alpha}$ = constant). Each of these cases may be summarized by writing $\underset{q}{\omega}^{\sigma} = \underset{m}{\omega}^{\sigma} = \omega^{\sigma}$ (say) and referring to ω^{σ} as the angular velocity of rotation of the particle, it being assumed that the electric and magnetic moments are being carried round with the particle at the same rate of rotation as the particle itself rotates. This is a fairly natural assumption to make and it may indeed be verified that the proper time derivative of $q^{\alpha}m_{\alpha}$ vanishes when $\underset{q}{\omega}^{\sigma} = \underset{m}{\omega}^{\sigma}$ from (2.8) and a similar equation for $\dot{m}^{\alpha}$. In these circumstances, from (2.2), equation (2.11) now becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) \equiv -\frac{1}{2c}\xi^{\alpha\beta\sigma\tau}C_{\alpha\beta}\omega_{\sigma}V_{\tau} = 0. \tag{2.12}$$

The rest of the discussion is now confined to the reduction of this equation. From (2.4) and (2.10) we have for the couple six-vector,

$$\begin{aligned} C_{\alpha\beta} &= \dot{S}_{\alpha\beta} - 2\dot{S}_{[\alpha\gamma} V^{\gamma} V_{\beta]} \\ &= c\{\xi_{\alpha\beta\mu\nu}(\dot{s}^{\mu} V^{\nu} + s^{\mu} \dot{V}^{\nu}) \\ &\quad - \xi_{\alpha\gamma\mu\nu}(\dot{s}^{\mu} V^{\nu} + s^{\mu} \dot{V}^{\nu}) V^{\gamma} V_{\beta} \\ &\quad + \xi_{\beta\gamma\mu\nu}(\dot{s}^{\mu} V^{\nu} + s^{\mu} \dot{V}^{\nu}) V^{\gamma} V_{\alpha}\}. \end{aligned} \tag{2.13}$$

Assuming now that the spin s^{μ} rotates with constant norm, in a like manner to q^{μ} and m^{μ}, so that we have an identical equation to (2.8) for $\dot{s}^{\mu}$:

$$\dot{s}^{\mu} = -(\dot{V}_{\lambda} s^{\lambda}) V^{\mu} + \frac{1}{c} \xi^{\mu\lambda\sigma\tau} s_{\lambda} \underset{s}{\omega}_{\sigma} V_{\tau} \tag{2.14}$$

where $\underset{s}{\omega}^{\sigma}$ represents the angular velocity of rotation of the spin angular momentum vector, we find that the Thomas term in (2.14) for s^{μ} again does not contribute to (2.13) and we are left with the equation

$$\begin{aligned} C_{\alpha\beta} = c\Big\{&\frac{1}{c} \xi_{\alpha\beta\mu\nu} \xi^{\mu\lambda\sigma\tau} V^{\nu} s_{\lambda} \underset{s}{\omega}_{\sigma} V_{\tau} \\ &+ \xi_{\alpha\beta\mu\nu} s^{\mu} \dot{V}^{\nu} - \xi_{\alpha\gamma\mu\nu} s^{\mu} \dot{V}^{\nu} V^{\gamma} V_{\beta} - \xi_{\gamma\beta\mu\nu} s^{\mu} \dot{V}^{\nu} V^{\gamma} V_{\alpha}\Big\}. \end{aligned}$$

The last three terms vanish identically (this can be established by calculating their duals) and we have

$$\begin{aligned} C_{\alpha\beta} &= \delta^{\lambda\sigma\tau}_{\nu\alpha\beta} V^{\nu} s_{\lambda} \underset{s}{\omega}_{\sigma} V_{\tau} \\ &= (\delta^{\lambda}_{\nu} \delta^{\sigma\tau}_{\alpha\beta} + \delta^{\sigma}_{\nu} \delta^{\tau\lambda}_{\alpha\beta} + \delta^{\tau}_{\nu} \delta^{\lambda\sigma}_{\alpha\beta}) V^{\nu} s_{\lambda} \underset{s}{\omega}_{\sigma} V_{\tau} \\ &= \delta^{\lambda\sigma}_{\alpha\beta} s_{\lambda} \underset{s}{\omega}_{\sigma} \\ &= -(\underset{s}{\omega}_{\alpha} s_{\beta} - \underset{s}{\omega}_{\beta} s_{\alpha}). \end{aligned} \tag{2.15}$$

In the instantaneous rest-frame, the nonzero components of $C_{\alpha\beta}$ are $(C_{23}, C_{31}, C_{12}) \equiv -\boldsymbol{C}$ where $\boldsymbol{C}$ is the electromagnetic couple, and the components of $\underset{s}{\omega}^{\alpha}$ and s^{α} are $\underset{s}{\omega}^{\alpha} \equiv (0, \underset{s}{\boldsymbol{\omega}})$, $s^{\alpha} \equiv (0, \boldsymbol{s})$ so that equation (2.15) is the covariant generalization of the equation $\boldsymbol{C} = \underset{s}{\boldsymbol{\omega}} \times \boldsymbol{s}$ in the instantaneous rest-frame.

Using (2.15), equation (2.12) now becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0 c^2) \equiv \frac{1}{c} \xi^{\alpha\beta\sigma\tau} \underset{s}{\omega}_{\alpha} s_{\beta} \omega_{\sigma} V_{\tau} = 0. \tag{2.16}$$

We show in an appendix† that this equation is equivalent to the equation

$$\begin{vmatrix} \underset{s}{\omega}_{\alpha} & \underset{s}{\omega}_{\beta} & \underset{s}{\omega}_{\sigma} \\ s_{\alpha} & s_{\beta} & s_{\sigma} \\ \omega_{\alpha} & \omega_{\beta} & \omega_{\sigma} \end{vmatrix} = 0 \tag{2.17}$$

† The author is indebted to James Foster for this proof.

which, in view of the complete antisymmetry, amounts to four separate equations, but in fact reduces to only one in the instantaneous rest-frame. The rows of this determinant are linearly dependent:

$$\omega_\alpha = \lambda s_\alpha + \mu \underset{s}{\omega}_\alpha \qquad (\alpha = 0, 1, 2, 3) \tag{2.18}$$

and this equation forms the general solution of (2.16) where λ and μ are some scalars. (In the event of the determinant having rank one, the vectors ω_α, s_α, $\underset{s}{\omega}_\alpha$ are proportional.)

However, (2.18) is in this case restricted by the requirement of constant spin. Multiplying (2.4) by $S_{\mu\nu}$ and using equations (2.10) we find

$$0 = -c^2 \frac{\mathrm{d}}{\mathrm{d}\tau}(s^\alpha s_\alpha) = S_{\mu\nu}\dot{S}^{\mu\nu} = S_{\mu\nu}C^{\mu\nu}.$$

Equation (2.2) therefore shows that

$$c\xi_{\mu\nu\sigma\tau}s^\sigma V^\tau(q^\mu F^{\nu\alpha} - m^\mu \overset{*}{F}{}^{\nu\alpha})V_\alpha = 0. \tag{2.19}$$

An immediate solution of (2.18) which is dynamically satisfactory and which is compatible with the equation (2.19) is

$$q_\alpha = \Lambda s_\alpha \qquad m_\alpha = \Gamma s_\alpha \tag{2.20}$$

where Λ, Γ are constants. In this case (2.19) is obviously satisfied and (2.18) is satisfied for $\lambda = 0, \mu = 1$. Another solution of (2.18), which is perhaps less interesting but which is compatible with (2.19) in certain circumstances, refers to particles, which include those we might term 'spherical type' particles, for which

$$s_\alpha = I\omega_\alpha$$

where I is some scalar (a moment of inertia). In this case (2.18) is satisfied for $\mu = 0$ and (2.19) is satisfied provided the dipole moments vary according to the equation

$$\dot{q}^\mu E_\mu + \dot{m}^\mu B_\mu = 0$$

(see (2.8), and equation (2.22) following).

It seems unlikely that other solutions, if they exist, would be as satisfactory as these two, and no other solutions have been found. It is apparent that the most acceptable solution to the equations in the case of constant spin and rest mass is when the spin and moment(s) are permanently in line.

We may conclude the discussion by confirming that equations (2.20) do in fact lead to constant rest mass, and in view of their significance it may be useful to add, as well, the case where Λ, Γ are allowed to vary. This leads to a simple relation between $\dot{m}_0$ and $\frac{1}{2}p^{\alpha\beta}F_{\alpha\beta}$ which is important and will be used in the next section.

Since $C_{\alpha\beta} = c\xi_{\alpha\beta\mu\nu}\dot{s}^\mu V^\nu$ (equation (2.13)), we have from the first equality in (2.5)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) = &-\tfrac{1}{2}\frac{\Lambda}{c}\overset{*}{F}_{\beta\gamma}C^{\beta\gamma} - \tfrac{1}{2}\frac{\Gamma}{c}F_{\beta\gamma}C^{\beta\gamma}\\ &-\tfrac{1}{2}\frac{\dot{\Lambda}}{c}\overset{*}{F}_{\beta\gamma}S^{\beta\gamma} - \tfrac{1}{2}\frac{\dot{\Gamma}}{c}F_{\beta\gamma}S^{\beta\gamma}\end{aligned} \tag{2.21}$$

where we have used (2.10). Without loss of generality we may represent by (1.1)

$$F_{\beta\gamma} = -(E_\beta V_\gamma - E_\gamma V_\beta) - \xi_{\beta\gamma\mu\nu} B^\mu V^\nu$$
$$F^*_{\beta\gamma} = (B_\beta V_\gamma - B_\gamma V_\beta) - \xi_{\beta\gamma\mu\nu} E^\mu V^\nu \tag{2.22}$$

where the electric and magnetic components E^μ, B^μ are orthogonal to V^μ. We find from (2.2) that

$$\tfrac{1}{2} F_{\beta\gamma} C^{\beta\gamma} = \xi_{\alpha\beta\sigma\tau} q^\alpha E^\beta B^\sigma V^\tau$$

$$\tfrac{1}{2} F^*_{\beta\gamma} C^{\beta\gamma} = -\xi_{\alpha\beta\sigma\tau} m^\alpha E^\beta B^\sigma V^\tau$$

and therefore the first two terms of (2.21) cancel. Hence

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0 c^2) = \tfrac{1}{2}\left\{(q^\mu V^\nu - q^\nu V^\mu)\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln \Lambda) + (\xi^{\mu\nu\alpha\beta} m_\alpha V_\beta)\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln \Gamma)\right\} F_{\mu\nu}.$$

If Λ, Γ are proportional, this equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0 c^2) = \tfrac{1}{2} p^{\mu\nu} F_{\mu\nu} \frac{\mathrm{d}}{\mathrm{d}\tau}(\ln \Gamma) \tag{2.23}$$

where $p^{\mu\nu}$ is the electromagnetic moment tensor

$$p^{\mu\nu} = q^\mu V^\nu - q^\nu V^\mu + \xi^{\mu\nu\alpha\beta} m_\alpha V_\beta. \tag{2.24}$$

We see at once that when Γ is constant, m_0 is constant. Other dynamical models of spin give rise to similar equations to equation (2.23) but in nearly all cases these will contain additional electromagnetic terms.

3. Comparison with a standard treatment based on the assumption of parallel spin and moment

It may be verified that the following expression for $\dot{S}^{\mu\nu}$ satisfies the equations of motion (2.2), (2.4)

$$\dot{S}^{\mu\nu} = -2p^{[\mu\alpha} F_\alpha{}^{\nu]} \tag{3.1}$$

where $p^{\mu\nu}$ is the electromagnetic dipole moment tensor (2.24). Expression (3.1) leads to a considerable simplification of the first equation of motion (2.1), (2.3) which we may see in the following way. From the expression for the mechanical momentum, we have from (3.1)

$$\begin{aligned} G^\mu &= m_0 c^2 V^\mu + 2p^{[\mu\alpha} F_\alpha{}^{\beta]} V_\beta \\ &= m_0 c^2 V^\mu - 2q_\alpha V^{[\mu} F^{\alpha\beta]} V_\beta + 2m_\alpha V^{[\mu} \overset{*}{F}{}^{\alpha\beta]} V_\beta \\ &= (m_0 c^2 - \tfrac{1}{2} p_{\alpha\beta} F^{\alpha\beta}) V^\mu + q_\alpha F^{\alpha\mu} - m_\alpha \overset{*}{F}{}^{\alpha\mu}. \end{aligned}$$

Differentiating this last expression and using the first equation of motion (2.1), (2.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{(m_0 c^2 - \tfrac{1}{2} p_{\alpha\beta} F^{\alpha\beta}) V^\mu\} = -eF^{\mu\nu} V_\nu - \tfrac{1}{2} p_{\alpha\beta} F^{\alpha\beta,\mu}$$

where we have used Maxwell's equations for $F^{\alpha\beta}$ in the last term. On completing the differentiation we find

$$(m_0c^2 - \tfrac{1}{2}p_{\alpha\beta}F^{\alpha\beta})\dot{V}^\mu = -eF^{\mu\nu}V_\nu - p^{\alpha\beta}F_{\alpha\beta,}{}^{[\mu}V^{\nu]}V_\nu. \tag{3.2}$$

The coefficient of the term in V^μ, which has been omitted from the above equation, must vanish on account of its being orthogonal to the other terms of (3.2); thus we have additionally,

$$\dot{m}_0c^2 = \tfrac{1}{2}\dot{p}_{\alpha\beta}F^{\alpha\beta}. \tag{3.3}$$

It will be seen that in the case where the fields are uniform ($F_{\alpha\beta,}{}^\mu \equiv 0$) equation (3.2) becomes identical with the equation of motion of a moving point charge

$$mc^2\dot{V}^\mu = -eF^{\mu\nu}V_\nu \tag{3.4}$$

where the mass of the particle is given by

$$m = m_0 - \frac{1}{2c^2}p_{\alpha\beta}F^{\alpha\beta}. \tag{3.5}$$

This is a constant by virtue of (3.3). (We note that in the absence of electromagnetic field the solution $S^{\mu\nu}$ = constant is compatible with a straight world line.)

Rohrlich (1965) gives an account of the same equation (3.1) as we have described above. However, his treatment differs from our own in that it is not related to a force equation containing dipole moment terms. He assumes that the conventional condition applies to the spin tensor and it is significant that he describes the equation (3.1) as the relativistic covariant generalization of the classical equation

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{m}\times\boldsymbol{B}$$

although it is apparent that, in an instantaneous rest-frame, the equations (3.1) for a magnetic dipole have the components

$$(\dot{S}_{23}, \dot{S}_{31}, \dot{S}_{12}) \equiv -\boldsymbol{m}\times\boldsymbol{B} \qquad (\dot{S}_{01}, \dot{S}_{02}, \dot{S}_{03}) \equiv \boldsymbol{m}\times\boldsymbol{E} \tag{3.6}$$

and an interpretation is lacking for the second three components. Nevertheless the consequences following from his treatment in the evaluation of the gyromagnetic ratio in the case of parallel moment and spin are valuable.

For this reason we feel it is important to try to remove these difficulties and to establish a similar line of reasoning leading to the value $g = 2$ for the spin gyromagnetic ratio.

Now we have placed an interpretation (in the case where $S^{\mu\nu}V_\nu = 0$) on all components of (3.6) since we have the covariant equations

$$c\xi_{\mu\nu\alpha\beta}\dot{s}^\alpha V^\beta = -(m_\mu B_\nu - m_\nu B_\mu) \qquad c(s^\mu\dot{V}^\nu - s^\nu\dot{V}^\mu) = m^\mu E^\nu - m^\nu E^\mu \tag{3.7}$$

from (3.1), and these equations are viewed as solutions of the more general equations (2.2), (2.4) where the couple tensor $C^{\mu\nu}$ is alone significant. It is clear that the solutions of equations (3.1) form only a special class of solutions of the equations (2.2), (2.4). However, they are the only solutions for which spin and dipole moment terms do not explicitly enter into the canonical momentum, as will be shown in § 5, where an application to quantum mechanics is studied. The equations (3.1) are themselves sufficient but not necessary since we may add to the right hand side of equations (3.1) any antisymmetric tensor of the form $C^\mu V^\nu - C^\nu V^\mu$, where C^μ is any space-like vector orthogonal to V^μ, and the equations for the couple (2.2), (2.4) remain satisfied. (The first equation of (3.7) is not affected by an addition of this kind and the spin will continue to be Thomas-precessed in the absence of couple.) Assuming that Rohrlich's equations (3.1) are satisfied, we see that they are sufficient for the determination of the six unknown values of s^μ and m^μ since they are six in number (see (3.7)). Therefore we may not in general impose the condition that s^μ and m^μ are parallel, as in Rohrlich's treatment, unless the second of the equations (3.7) is satisfied. In the case where the fields are uniform this latter equation is satisfied identically, and this is a case we now discuss.

Evaluation of the gyromagnetic ratio in the case of uniform field $F^{\mu\nu,\alpha} \equiv 0$

It is instructive to set up a solution involving the quantities E^μ, B^μ, measured along the worldline. Considering E^μ and B^μ as functions of proper time along the worldline of the motion $z^\mu = z^\mu(\tau)$, we have on differentiating (2.22):

$$0 = F^{\mu\nu,\alpha}\dot{z}_\alpha = -(\dot{E}^\mu V^\nu - \dot{E}^\nu V^\mu) + \xi^{\mu\nu\alpha\beta}\dot{B}_\alpha V_\beta - (E^\mu \dot{V}^\nu - E^\nu \dot{V}^\mu) + \xi^{\mu\nu\alpha\beta} B_\alpha \dot{V}_\beta.$$

From this it follows that

$$\begin{aligned} \dot{E}^\mu &= (\dot{E}^\alpha V_\alpha) V^\mu + \xi^{\mu\nu\alpha\beta} V_\nu B_\alpha \dot{V}_\beta \\ \dot{B}^\mu &= (\dot{B}^\alpha V_\alpha) V^\mu - \xi^{\mu\nu\alpha\beta} V_\nu E_\alpha \dot{V}_\beta. \end{aligned} \tag{3.8}$$

Since the field is uniform it follows from (3.4) that E^μ is parallel to $\dot{V}^\mu$, and therefore (3.8) gives immediately

$$\dot{B}^\mu = (\dot{B}^\alpha V_\alpha) V^\mu \qquad \dot{B}^\mu B_\mu = 0 \qquad \dot{E}^\mu E_\mu = 0 \tag{3.9}$$

and E^μ and B^μ have constant norm, measured along the worldline, and B^μ does not rotate (Fermi–Walker propagation). Since E^μ has constant norm it follows again from equation (3.4) that the acceleration of the particle is uniform:

$$\dot{V}^\mu \dot{V}_\mu = \text{constant}.$$

In these circumstances it is clear that the second equation of (3.7) may be satisfied by taking m^μ and s^μ parallel, and connected by

$$m^\mu = \Gamma s^\mu \qquad \Gamma = e/mc$$

where m is the 'modified' mass (3.5). From the work of § 2, since Γ is constant, s^μ (and therefore m^μ) must have constant norm, and the mass m_0 is also constant. It therefore follows that the dipole potential energy $V = -\frac{1}{2}p_{\alpha\beta}F^{\alpha\beta} = m^\mu B_\mu$ is a constant of the motion, as may independently be verified from the equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(s^\mu B_\mu) = \dot{s}^\mu B_\mu = 0$$

which follow immediately from (3.9) and the dual of the first equation of (3.7). (Although $s^\mu B_\mu$ = constant and B^μ does not rotate, it does not follow from this that s^μ does not rotate, since we have the equation

$$c\dot{s}^\mu = c(\dot{s}^\alpha V_\alpha) V^\mu - \xi^{\mu\nu\alpha\beta} V_\nu m_\alpha B_\beta$$

and the only occasion in which this happens is in the absence of couple.)

We therefore have a fully consistent picture in the case of a uniform field, in which the motion of the particle, and the position of the moment and spin, are known at any particular instant. It leads to a value for the gyromagnetic ratio g through the expression

$$\frac{ge}{2m_0c} \equiv \Gamma = \frac{e}{\left(m_0 - \frac{1}{2c^2}p_{\alpha\beta}F^{\alpha\beta}\right)c} \tag{3.10}$$

and is slightly different from the usual value $g = 2$ as obtained by Rohrlich.

There is no *a priori* reason to suppose that (3.10) is an exact relationship which remains valid in the presence of non-uniform fields. For equations (3.1) to hold we may not in general suppose that the spin and dipole moment remain parallel. However, if, in the context of the previous section, we make the choice (3.10) for Γ, we find that equation (2.23) may be integrated:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) = -\frac{\dot{m}}{2m}p^{\mu\nu}F_{\mu\nu} \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}(mc^2) = -\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}\tau}(mp^{\mu\nu}F_{\mu\nu}).$$

This gives

$$m^2c^4 + mc^2p^{\mu\nu}F_{\mu\nu} = (m_0c^2)^2 - (\tfrac{1}{2}p^{\mu\nu}F_{\mu\nu})^2 = \text{constant}.$$

The change in dipole potential $V = -\frac{1}{2}p^{\mu\nu}F_{\mu\nu}$ of the particle must therefore be related to its change in mechanical mass m_0 by the equation

$$\Delta V = c^4\frac{m_0\delta m_0}{V}$$

to first order. This expression is reminiscent of the role played in mass renormalization.

4. Variable spin

Considering now the equations (2.2), (2.4) in their general form, we suppose that the electric and magnetic dipole moments and the spin are given by the equations

$$q^\mu = q\hat{q}^\mu \qquad m^\mu = m\hat{m}^\mu \qquad s^\mu = s\hat{s}^\mu$$

where the norms q, m, s representing the magnitudes of the moments and the spin are allowed to vary. The unit directions $\hat{q}^\mu$, $\hat{m}^\mu$, $\hat{s}^\mu$ satisfy the equations

$$\hat{q}^\mu\hat{q}_\mu = \hat{m}^\mu\hat{m}_\mu = \hat{s}^\mu\hat{s}_\mu = -1$$

and we suppose that the vectors $\hat{q}^\mu$, $\hat{m}^\mu$, $\hat{s}^\mu$ have rates of rotation given by the previous angular velocity vectors (of § 2) ω^μ, ω^μ, $\underset{s}{\omega}^\mu$ respectively. Hence in comparison with (2.8) and (2.14) we have

$$\begin{aligned}\dot{q}^\alpha &= \dot{q}\hat{q}^\alpha + q\dot{\hat{q}}^\alpha \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln q)\right\}q^\alpha - (\dot{V}_\beta q^\beta)V^\alpha + \frac{1}{c}\xi^{\alpha\beta\mu\nu}q_\beta\omega_\mu V_\nu\end{aligned} \tag{4.1}$$

and similar equations for $\dot{m}^\alpha$ and $\dot{s}^\mu$. Equation (2.5) therefore becomes, with the help of (2.9) with q^μ and m^μ replaced by unit vectors,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) = -\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln q)\right\}q^\alpha E_\alpha - \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln m)\right\}m^\alpha B_\alpha - \frac{1}{2c}\xi^{\alpha\beta\sigma\tau}C_{\alpha\beta}\omega_\sigma V_\tau \tag{4.2}$$

comparing with equation (2.12).

Now from a similar equation to (4.1) for $\dot{s}^\mu$ we find

$$C_{\alpha\beta} = \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln s)\right\} S_{\alpha\beta} - 2\underset{s}{\omega}_{[\alpha} s_{\beta]}$$

corresponding to (2.15). Hence (4.2) finally becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0c^2) &\equiv \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln m_0)\right\} m_0c^2 \\ &= -\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln q)\right\} q^\alpha E_\alpha - \left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln m)\right\} m^\alpha B_\alpha \\ &\quad -\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln s)\right\} s^\alpha \omega_\alpha + \frac{1}{c}\,\xi^{\alpha\beta\sigma\tau} \underset{s}{\omega}_\alpha s_\beta \omega_\sigma V_\tau. \end{aligned} \tag{4.3}$$

In this equation the quantities m_0c^2, $q^\alpha E_\alpha$, $m^\alpha B_\alpha$, $-\frac{1}{2}s^\alpha\omega_\alpha$ represent the rest energy, the dipole potential energies and the classical spin energy of the particle respectively The equation (4.3) reduces to equation (2.23) when s^μ and m^μ are parallel and q^μ is zero, for in this case $\dot{}$ $= 0$. The other consequence of equation (4.3) is the case of 'spherical'-type particles where the spin angular momentum s^μ and the angular velocity ω^μ of the particle are parallel. If we assume that the rest mass m_0 and the magnitude of the spin s increase in the same proportion, the energy equation

$$m_0c^2 = -s^\alpha\omega_\alpha \qquad (= -\tfrac{1}{2}S_{\alpha\beta}\Omega^{\alpha\beta})$$

is satisfied identically for particles of this type when the dipole moments have constant strength. (If the moments themselves increase at the same rate, this energy equation is modified to $mc^2 = -s^\alpha\omega_\alpha$ where m is given by (3.5).)

5. Application to quantum mechanics

From the Lagrangian for a single particle possessing charge and dipole moments interacting with an external field, we have the canonical momenta†

$$p_\mu = -\frac{\partial L}{\partial V^\mu} \qquad S_{\mu\nu} = -2\frac{\partial L}{\partial \dot{q}^{[\mu}{}_{(\alpha)}} q_{\nu](\alpha)}.$$

For the first of these we have the expression

$$p^\mu = G^\mu + eA^\mu + q_\alpha F^{\mu\alpha} - m_\alpha \overset{*}{F}{}^{\mu\alpha}$$

which may be cast in the form

$$\begin{aligned} p^\mu &= (m_0c^2 - \tfrac{1}{2}p_{\alpha\beta}F^{\alpha\beta})V^\mu + eA^\mu \\ &\quad - \{\dot{S}^{\mu\nu} - \xi^{\mu\alpha\beta\nu}(q_\alpha B_\beta - m_\alpha E_\beta)\}V_\nu. \end{aligned}$$

Now it can be shown that this reduces to the familiar form

$$p^\mu = mc^2V^\mu + eA^\mu \tag{5.1}$$

† The sign of the Lagrangian should be reversed from that given in the previous article (Ellis 1970).

in the case of Rohrlich's equations (3.1), m being given by (3.5). This therefore illustrates the significance of Rohrlich's equations. Following (5.1) we have the equation

$$(p^{\mu}-eA^{\mu})(p_{\mu}-eA_{\mu}) = m^2c^4. \tag{5.2}$$

Taking for the energy momentum operator

$$p^{\mu} = (E, \mathbf{p}c) = i\hbar c\partial^{\mu}$$

we may obtain the Klein–Gordan (or Dirac) equations with potentials from (5.2):

$$\left[\left(\partial^{\mu}+\frac{ieA^{\mu}}{\hbar c}\right)^2+\frac{m^2c^2}{\hbar^2}\right]\psi = 0.$$

From the second invariant

$$\tfrac{1}{2}S^{\mu\nu}S_{\mu\nu} = c^2s^2$$

the usual procedure can be applied in the case of zero field (but presumably also in the case of nonzero field) to determine the spin states (see, for example, Corson 1953 p. 95).

Appendix

We here show that (2.16) is equivalent to (2.17). Write (2.16) in the form

$$0 = \xi^{\alpha\beta\sigma\tau}H_{\alpha\beta\sigma}V_{\tau}$$

where

$$H_{\alpha\beta\sigma} = \tfrac{1}{6}(\underset{s}{\omega_{\alpha}}s_{\beta}\omega_{\sigma}+\underset{s}{\omega_{\beta}}s_{\sigma}\omega_{\alpha}+\underset{s}{\omega_{\sigma}}s_{\alpha}\omega_{\beta}-\underset{s}{\omega_{\alpha}}s_{\sigma}\omega_{\beta}-\underset{s}{\omega_{\sigma}}s_{\beta}\omega_{\alpha}-\underset{s}{\omega_{\beta}}s_{\alpha}\omega_{\sigma})$$

and take the orthonormal Vierbein

$$\overset{(0)}{e_{\mu}}(=V_{\mu}),\ \overset{(1)}{e_{\mu}},\ \overset{(2)}{e_{\mu}},\ \overset{(3)}{e_{\mu}}$$

where the suffix μ is tensorial. Then

$$H_{\alpha\beta\sigma} = H_{(bcd)}\overset{(b)}{e_{\alpha}}\overset{(c)}{e_{\beta}}\overset{(d)}{e_{\sigma}}$$

where b, c, d range over the values 0, 1, 2, 3. Since $H_{\alpha\beta\sigma}$ is completely antisymmetrical, it follows that $H_{(bcd)}$ is completely antisymmetrical. Substitution yields

$$0 = \xi^{\alpha\beta\sigma\tau}\overset{(b)}{e_{\alpha}}\overset{(c)}{e_{\beta}}\overset{(d)}{e_{\sigma}}\overset{(0)}{e_{\tau}}H_{(bcd)}$$

where b, c, d now range over the unequal values 1, 2, 3. Hence

$$H_{(123)} = 0.$$

From this it follows that $H_{\alpha\beta\sigma}$ is of the form

$$H_{\alpha\beta\sigma} = P_{\alpha\beta}V_{\sigma}+P_{\beta\sigma}V_{\alpha}+P_{\sigma\alpha}V_{\beta}$$

where $P_{\alpha\beta} = H_{(0bc)}e^{(b)}_{\alpha}e^{(c)}_{\beta}$ is antisymmetric. But since $H_{\alpha\beta\sigma}V^{\sigma} = 0$, $P_{\alpha\beta}$ has the form

$$P_{\alpha\beta} = \lambda_\alpha V_\beta - \lambda_\beta V_\alpha$$

and so $H_{\alpha\beta\sigma}$ is identically zero. Equation (2.17) then follows.

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